OP, OQ, and OR, we have

$$OP^2(\cos 2\alpha + i \sin 2\alpha) + OQ^2(\cos 2\alpha' + i \sin 2\alpha') = - OR^2(\cos 2\alpha'' + i \sin 2\alpha''),$$

from datum; hence,

$$OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = - OR^2 \cos 2\alpha'' \tag{1}$$

and

$$OP^2 \sin 2\alpha + OQ^2 \sin 2\alpha' = - OR^2 \sin 2\alpha''. \tag{2}$$

Now P and Q being points on the ellipse, we have from known properties,

$$OP^2 \cos^2 \alpha + OQ^2 \cos^2 \alpha' = a^2, \qquad OP^2 \sin^2 \alpha + OQ^2 \sin^2 \alpha' = b^2;$$

hence, $OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = a^2 - b^2$, and (1) becomes

$$- OR^2 \cos 2\alpha'' = a^2 - b^2. \tag{3}$$

Also, from known properties concerning the ends of conjugate diameters,

$$OP^2 \sin 2\alpha = - OQ^2 \sin 2\alpha';$$

hence, (2) becomes

$$- OR^2 \sin 2\alpha'' = 0. \tag{4}$$

It follows from (3) and (4), that $2\alpha'' = 180°$ or $540°$, and $OR^2 = a^2 - b^2$, that is, $OR = \sqrt{a^2 - b^2}$, the distance from the center to focus, and $\alpha'' = 90°$ or $270°$, which shows that OR lies on the minor axis.

Also solved by H. Halperin, A. M. Harding, and H. L. Olson.

2780 [1919, 311]. Proposed by Elmer Latshaw, West Philadelphia, Pa.

A quadrilateral whose sides are a, $2a$, $3a$, $4a$ is inscribed in a circle. Find the radius of the circle.

I. Solution by H. S. Uhler, Yale University.

The interest in this problem may be enhanced by giving a perfectly general solution. Let the sides of any convex inscriptible quadrilateral be denoted by a_1, a_2, a_3, a_4. A diagonal c may be drawn dividing the quadrilateral into two non-overlapping triangles the sides of which are a_1, a_2, c and a_3, a_4, c, respectively. If the angle between a_1 and a_2 be symbolized by C, the angle between a_3 and a_4 must be $180° - C$. Accordingly

$$c^2 = a_1^2 + a_2^2 - 2a_1a_2 \cos C,$$

$$c^2 = a_3^2 + a_4^2 + 2a_3a_4 \cos C.$$

Eliminating $2\cos C$ we find

$$c^2 = \frac{(a_1a_3 + a_2a_4)(a_2a_3 + a_4a_1)}{a_1a_2 + a_3a_4}. \tag{1}$$

The area of a plane triangle having the sides a_1, a_2, c is given by either member of the following equation

$$\frac{a_1a_2c}{4R} = \sqrt{s(s - a_1)(s - a_2)(s - c)}, \tag{2}$$

where $2s = a_1 + a_2 + c$, and R denotes the radius of the circumscribed circle.

Substituting the trinomial value of s in equation (2) we obtain

$$\frac{a_1a_2c}{R} = \sqrt{[(a_1 + a_2)^2 - c^2][c^2 - (a_1 - a_2)^2]}. \tag{3}$$

Replacing c in equation (3) by expression (1) we eventually find that

$$R = \frac{\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}}{\sqrt{(a_2 + a_3 + a_4 - a_1)(a_3 + a_4 + a_1 - a_2)(a_4 + a_1 + a_2 - a_3)(a_1 + a_2 + a_3 - a_4)}}, \tag{4}$$

or

$$R = \frac{1}{4K}\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}. \tag{5}$$

where if $2S = a_1 + a_2 + a_3 + a_4$, $K = \sqrt{(S - a_1)(S - a_2)(S - a_3)(S - a_4)}$ = area of quadrilateral.

The denominator of formula (4) brings out the geometrically-evident fact that each side of the quadrilateral must not exceed the sum of the remaining three sides. When the quadrilateral degenerates into a straight line, formulas (4) and (5) give $R = \infty$, as they should. These formulæ also show explicitly that the order or succession of the sides has no effect on the value of R, a fact which is obvious geometrically since the sum of the arcs subtended by the four sides of the quadrilateral equals the entire circumference.

The answer to the given problem may be obtained at once from formula (5) by substituting a, $2a$, $3a$, $4a$, $5a$ for a_1, a_2, a_3, a_4, S respectively. It is

$$R = \frac{a\sqrt{385}}{4\sqrt{6}} = (2{\cdot}002602\cdots)a.$$

II. Solution by Bing Chin Wong, Berkeley, Calif.

Let $ABCD$ be the polygon with sides $AB = a$, $BC = 2a$, $CD = 3a$, $DA = 4a$ inscribed in the circle with O as center. Join O to A, B, C, D. Then

$$\angle AOB + \angle BOC + \angle COD + \angle DOA = 2\alpha + 2\beta + 2\gamma + 2\delta = 2\pi,$$

or

$$\alpha + \beta + \gamma + \delta = \pi.$$

Then

$$\cos(\alpha + \beta) + \cos(\gamma + \delta) = 0,$$

or

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\gamma\cos\delta - \sin\gamma\sin\delta = 0. \tag{I}$$

Let r be the radius of the circle. We obtain from the figure,

$$\sin\alpha = \frac{a}{2r}, \qquad \sin\beta = \frac{a}{r}, \qquad \sin\gamma = \frac{3a}{2r}, \qquad \sin\delta = \frac{2a}{r};$$

and

$$\cos\alpha = \frac{\sqrt{4r^2 - a^2}}{2r}, \qquad \cos\beta = \frac{\sqrt{r^2 - a^2}}{r}, \qquad \cos\gamma = \frac{\sqrt{4r^2 - 9a^2}}{2r}, \qquad \cos\delta = \frac{\sqrt{r^2 - 4a^2}}{r}.$$

Substituting these values in (I) and multiplying by $2r^2$, we have

$$\sqrt{4r^2 - a^2}\sqrt{r^2 - a^2} + \sqrt{4r^2 - 9a^2}\sqrt{r^2 - 4a^2} = 7a^2.$$

Squaring, collecting terms, and dividing by 2, we have

$$\sqrt{(4r^2 - a^2)(r^2 - a^2)(r^2 - 4a^2)(4r^2 - 9a^2)} = 6a^4 + 15a^2r^2 - 4r^4.$$

Squaring again and collecting terms, we have

$$96a^4r^4 = 385a^6r^2, \qquad \text{or} \qquad r^2 = 385a^2/96,$$

and, therefore,

$$r = a\sqrt{385/96} = a\sqrt{2310}/24 = 2.0026a.$$

Also solved by H. C. Bradley, H. N. Carleton, S. A. Corey, Laura Guggenbuhl, T. F. Noismann, H. L. Olson, A. Pelletier, J. B. Reynolds, and the Proposer.

2781 [1919, 311]. Proposed by J. L. Riley, Stephensville, Texas.

Show that the asymptotic lines on a pseudospherical surface are curves of constant torsion.[1]

Solution by Otto Dunkel, Washington University.

The Theorem of Euneper states that the square of the torsion of an asymptotic line at any point of a surface is equal to the total curvature of the surface (with sign changed) at the point (Eisenhart, *Differential Geometry*, p. 140). Since it is *known* that the total curvature of a pseudospherical surface is constant, it follows at once that the torsion is constant.

[1] This problem is given as an example in Eisenhart's *Differential Geometry*, p. 290.—Editors.